Solving Eq. (3.10) for h, we obtain

$$h = -\frac{d\gamma}{3am^2} c + c^3(\ldots)$$

Substituting the resulting value of h into formula (3.9) and the resulting expression for y° into formula (3.3), we obtain the first approximation of the required periodic solution of system (3.2). The formula

$$x_t^\circ(\vartheta) = z_{1t}^\circ(\vartheta) + b(\vartheta)y^\circ + \bar{b}(\vartheta)\bar{y}^\circ$$

yields the first approximation of the periodic solution of system (3.1). Computation of the subsequent approximation is not difficult. The expressions involved are extremely cumbersome, however.

BIBLIOGRAPHY

1. Malkin, I. G., Some Problems of the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
2. Krasovskii, N. N., Some Problems of the Theory of Motion Stability. Moscow, Fizmatgiz, 1959.
3. Shimanov, S. N., The critical case of a pair of purely imaginary roots for systems with lag. Sib. Mat. Zhurnal Vol. 2, №3, 1961.
4. Kleimenov, A. F. and Shimanov, S. N., On the existence and construction of periodic solutions of systems with lag closely related to Liapunov systems. Differentsial'nye Uravneniia Vol. 4, №7, 1968.

Translated by A. Y.

A STUDY OF NONLINEAR SYSTEMS OSCILLATIONS

PMM Vol. 33, №3, 1969, pp. 413-430
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(Received October 22, 1968)

A procedure for investigating oscillations based on the small parameter method is described. The proposed procedure involves the use of nonlinear difference equations of special form. A mathematical justification of the procedure will be found in [1]. It consists essentially in the construction of an ancillary system of differential equations whose solution coincides at certain instants with the solution of the initial system. Applications considered include cases of resonance in quasilinear systems. A first-approximation integral stability criterion for periodic and almost-periodic solutions is derived.

1. The difference equations. Let us consider the following system of difference equations of order m:

$$X_{n+1} - X_n = \mu\Psi(X_n, n\mu, \mu) \qquad (n = 0, 1, 2, \ldots) \tag{1.1}$$

We assume that the right side is differentiable a sufficient number of times with respect to all of its arguments in some domain containing the solution X_n. We also assume that the parameter μ is small and that $\mu \geqslant 0$. Let us turn from (1.1) to a more general system of difference equations, introducing the ancillary vector function $Z(\tau, \mu)$ such that

$$Z(n\mu, \mu) = X_n \qquad (n = 0, 1, 2, \ldots) \tag{1.2}$$

System (1.1) can be written as

$$Z(\tau+\mu,\mu)-Z(\tau,\mu)=\mu\,\Psi(\tau,Z(\tau,\mu),\mu),\quad \tau=n\mu \tag{1.3}$$
$$\Psi(\tau,Z,\mu)=\Psi_0(\tau,Z)+\mu\,\Psi_1(\tau,Z)+\mu^2\Psi_2(\tau,Z)+\ldots$$

In order to be able to use the conventional methods of mathematical analysis, we assume that τ is a continuously varying argument. In addition, we require that system of difference equations (1.3) be satisfied not only for discrete values of $\tau=n\mu$ ($n = 0, 1, 2, \ldots$), but also for all values of τ. As is shown in [1], system (1.3) has a unique solution $Z(\tau,\mu)$ which can be expanded in a series asymptotic as $\mu\to 0$,

$$Z(\tau,\mu)=Z_0(\tau)+\mu Z_1(\tau)+\mu^2 Z_2(\tau)+\ldots,\ Z(0,\mu)=X_0 \tag{1.4}$$

Series (1.4) generally diverges, but has the following asymptotic property:

$$\left|Z(\tau,\mu)-\sum_{j=0}^{k}Z_j(\tau)\mu^j\right|=O(\mu^{k+1}),\quad \mu\to 0 \tag{1.5}$$

The asymptotic character of expansion (1.4) is proved in [1], p. 969. The actual construction of $Z(\tau,\mu)$ can be effected by the small-parameter method, i.e. by substituting (1.4) into (1.3) and equating the coefficients of the expansion in powers of the parameter μ. This yields the following system of differential equations for $Z_0(\tau)$:

$$\frac{dZ_0}{d\tau}=\Psi(\tau,Z_0,0),\qquad Z_0(0)=X_0 \tag{1.6}$$

If the general solution can be found for system (1.6), then all the vectors $Z_j(\tau)$ can be determined successively for an arbitrary initial vector X_0 [1].

2. The ancillary system of differential equations. System (1.6) is not integrable in the general case. In order to able to apply qualitative methods we must find a system of differential equations

$$\frac{dZ}{d\tau}=S(\tau,Z,\mu),\qquad Z=X_0,\qquad \tau=0 \tag{2.1}$$

satisfied by the solution $Z(\tau,\mu)$ of system of difference equations (1.3). We call system (2.1) the "ancillary system of differential equations". Construction of the vector $S(\tau,Z,\mu)$ from a given vector $\Psi(\tau,Z,\mu)$ is exceedingly difficult because of the complex analytic structure of $S(\tau,Z,\mu)$. Taking a first-order difference equation as an example, we shall show that despite the analyticity of the function $\Psi(\tau,Z,\mu)$, the function $S(\tau,Z,\mu)$ has a singularity (a pole condensation point) for $\mu=0$.

Example 2.1. The solution $z(\tau,\mu)$ of the difference equation

$$z(\tau+\mu,\mu)-z(\tau,\mu)=-\mu(\tau+1)^{-2}$$

is given by a series which converges together with its derivatives for $\tau\geqslant 0$, namely by

$$z(\tau,\mu)=\frac{\mu}{(\tau+1)^2}+\frac{\mu}{(\tau+1+\mu)^2}+\frac{\mu}{(\tau+2+\mu)^2}+\ldots$$

For a fixed τ the function $z(\tau,\mu)$ has poles at the points $\mu=-(\tau+1)n^{-1}$, so that the radius of convergence of the expansion of $z(\tau,\mu)$ in powers of μ is equal to zero. It is therefore necessary to seek forms of the solution which differ from asymptotic expansion. Solution (1.4) depends on the m parameter-coordinates of the vector X_0. Eliminating these coordinates, we arrive at system of differential equations (2.1) where

$$S(\tau,Z,\mu)=S_0(\tau,Z)+\mu S_1(\tau,Z)+\mu^2 S_2(\tau,Z)+\ldots \tag{2.2}$$

The vectors $S_j\ (\tau, Z)$ can be readily determined from the known vectors $\Psi_j\ (\tau, Z)$ by differentiation. This yields (for the derivation see, e.g. [1]),

$$S_0\ (\tau, Z) = \Psi_0\ (\tau, Z) \tag{2.3}$$

$$S_j\ (\tau, Z)_j' + G_j\ (S_0, S_1, \ldots, S_{j-1}) = \Psi_j'\ (\tau, Z) \quad (j = 1,2,3,\ldots)$$

Here the projections G_j are polynomials in the projections of the already computed vectors $S_0\ (\tau, Z), \ldots, S_{j-1}\ (\tau, Z)$ and their derivatives. Expansion (2.2) generally converges because of the singularity at $\mu = 0$. This obliges us to seek other forms of analytic representation of the vector $S\ (\tau, Z, \mu)$.

In particular, there is a way of constructing ancillary system (2.1) with the aid of the symbolic operators Δ, d. We can write

$$S\,(\tau, Z, \mu) = \Psi\,(\tau, Z, \mu) - \frac{\Delta}{2}\,\Psi\,(\tau, Z, \mu) + \frac{\Delta^2}{3}\,\Psi\,(\tau, Z, \mu) - \ldots \tag{2.4}$$

The expressions for the $\Delta^k\Psi$ are computed from system of difference equations (1.3) and are of the form

$$\Phi_{k+1}\ (\tau, Z, \mu) = \Phi_k\ (\tau + \mu, \quad Z + \mu\Psi\ (\tau, Z, \mu), \mu) - \Phi_k\ (\tau, Z, \mu) \tag{2.5}$$

$$\Phi_k\ (\tau, Z, \mu) \equiv \Delta^k\Psi\ (\tau, Z, \mu)$$

Conversely, we have the following expression for Ψ:

$$\Psi\,(\tau, Z, \mu) = S\,(\tau, Z, \mu) + \frac{\mu}{2!}\,dS\,(\tau, Z, \mu) + \frac{\mu^2}{3!}\,d^2S\,(\tau, Z, \mu) + \ldots \tag{2.6}$$

where d is the differentiation operator by virtue of system (2.1),

$$d^{k+1}S\,(\tau, Z, \mu) = \frac{Dd^kS\,(\tau, Z, \mu)}{DZ}\,S\,(\tau, Z, \mu) \tag{2.7}$$

3. The method of successive substitutions. System of difference equations (1.3) can be solved by a system of successive substitutions similar to the method of variation of arbitrary constants. Specifically, we solve the system

$$\frac{dZ}{d\tau} = \Psi\,(\tau, Z, \mu) \tag{3.1}$$

representing the solution in the form of a system of integrals,

$$C = \Pi\ (\tau,\ Z\,(\tau),\ \mu), \qquad C = \text{const} \tag{3.2}$$

Here and below we shall refrain from writing μ as an argument of y unknowns. The above system can be solved for $Z\,(\tau)$,

$$Z\,(\tau) = \Theta\,(\tau, C, \mu), \quad \frac{\partial\Theta\,(\tau, C, \mu)}{\partial\tau} \equiv \Psi\,(\tau, \Theta\,(\tau, C, \mu), \mu) \tag{3.3}$$

To be specific, as our C we can take the initial vector in such a way that

$$C \equiv \Pi\ (0, C, \mu) \tag{3.4}$$

We now replace the constant vector C by the variable vector Y. We shall attempt to find the solution of system (1.3) in the form

$$Z\,(\tau) = \Theta\,(\tau, Y\,(\tau), \mu) \tag{3.5}$$

where $Y\,(\tau)$ is the new unknown vector. Substituting this into (1.3) and expanding in powers of μ, we find that

$$\mu\,\frac{\partial\Theta\,(\tau, Y\,(\tau), \mu)}{\partial\tau} + \frac{D\Theta\,(\tau, Y\,(\tau), \mu)}{DY\,(\tau)}\,[Y\,(\tau+\mu) - Y\,(\tau)] = \mu\Psi\,(\tau, \Theta\,(\tau, Y\,(\tau), \mu)) + O\,(\mu^2) \tag{3.6}$$

By virtue of the second condition of (3.3), we find from Eq. (3.6) that

$$Y(\tau+\mu) - Y(\tau) = O(\mu^2) \tag{3.7}$$

However, it is more convenient to construct the system of difference equations for $Y(\tau)$ with the aid of the following formula for $Y(\tau)$:

$$Y(\tau) = \Pi(\tau, Z(\tau), \mu) \tag{3.8}$$

In this way we arrive at the system of difference equations

$$Y(\tau+\mu) - Y(\tau) = \Pi(\tau+\mu, Z(\tau) + \mu\Psi(\tau, Z(\tau), \mu), \mu) - \Pi(\tau, Z(\tau), \mu) \tag{3.9}$$

Next, eliminating $Z(\tau)$ from the right side with the aid of (3.5) and recalling (3.7), we finally obtain an equation of the form

$$Y(\tau+\mu) - Y(\tau) = \mu^2\Omega(\tau, Y(\tau), \mu) \tag{3.10}$$

We then repeat the entire procedure the required number of times. The order of the right-hand side with respect to μ doubles with each substitution. The possibility of actually effecting the substitutions depends on the possibility of solving system (3.1) in general form, since with subsequent substitutions the systems of differential equations are readily integrable by the asymptotic method. Thus, to find the substitution for Y we must solve the system of equations

$$\frac{dY}{d\tau} = \mu\Omega(\tau, Y, \mu) \tag{3.11}$$

In solving system (3.11) we can alter the terms of order μ in the right side. In system (3.11) we can alter the terms of order μ^3 and higher, etc. We can exploit this freedom of action in integrating systems (3.1), (3.11), ... We note that the method of substitutions is similar to that used by Kolmogorov and Arnol'd to solve differential equations [2].

Example 3.1. Let us find an approximate solution of the difference equation

$$z(\tau+\mu) - z(\tau) = \mu z^2(\tau) \tag{3.12}$$

Solving the ancillary differential equation

$$\frac{dz}{d\tau} = z^2, \qquad c = \frac{z}{1+z\tau}, \qquad z = \frac{c}{1-c\tau}$$

we obtain an approximate solution for z. We then make the following substitutions in the difference equation:

$$z(\tau) = \frac{y(\tau)}{1-\tau y(\tau)}, \qquad y(\tau) = \frac{z(\tau)}{1+\tau z(\tau)} \tag{3.13}$$

This gives us our difference equation for $y(\tau)$,

$$y(\tau+\mu) - y(\tau) = \mu^2 y^2(\tau)\ [1 + (\mu-\tau)y(\tau) + \mu^2 y^2(\tau)]^{-1}$$

Solving the ancillary differential equation

$$\frac{dy}{d\tau} = \mu\frac{y^3}{1+y(\mu-\tau)+\mu^2y^2}$$

we obtain an approximate expression for $y(\tau)$,

$$y(\tau) = c + \mu c^2 \ln(1-c\tau) + \mu^2c^3\left[\ln^2(1-c\tau) + \frac{\ln(1+c\tau)+2}{1+c\tau}\right] + O(\mu^3)$$

Substituting $y(\tau)$ into $z(\tau)$ (3.13), we obtain a solution of Eq. (3.12) accurate to within terms of order μ^2 , inclusive.

4. Asymptotic integration of essentially nonlinear oscillating systems. Let us consider the following system of equations with the small parameter μ :

$$\frac{dX}{dt} = F(t, X, \mu), \qquad F(t+2\pi, X, \mu) \equiv F(t, X, \mu) \tag{4.1}$$

We assume that the solutions of system (4.1) are extendible in t, and that the right side of $F(t, X, \mu)$ is differentiable the required number of times with respect to all of its arguments in a sufficiently large domain. We also assume that the general solution of the system

$$X = \Phi(t, X_0, \mu), \qquad \Phi(0, X_0, \mu) = X_0 \tag{4.2}$$

satisfies the condition

$$\Phi(t + 2\pi, X_0, 0) \equiv \Phi(t, X_0, 0) \tag{4.3}$$

i.e. that all the solutions of system (4.1) become 2π-periodic for $\mu = 0$. An example of such a system is the system in standard form

$$\frac{dX}{dt} = \mu F(t, X, \mu), \qquad F(t+2\pi, X, \mu) \equiv F(t, X, \mu) \tag{4.4}$$

which is readily amenable to the use of asymptotic methods [3]. All of its solutions are constant and satisfy condition (4.3) for $\mu = 0$.

Let us introduce the notation

$$X_n = \Phi(2\pi n, X_0, \mu) \tag{4.5}$$

By virtue of the periodicity of the right side of the system (4.1), we have

$$X_{n+1} = \Phi(2\pi, X_n, \mu), \qquad \Phi(2\pi, X_n, \mu) = X_n + O(\mu) \tag{4.6}$$

Finally, we can write the system of difference equations relating X_n and X_{n+1} as

$$X_{n+1} - X_n = \mu\Psi(X_n, \mu) \qquad (n = 0,1,2, \ldots) \tag{4.7}$$

$$\mu\Psi(X_n, \mu) \equiv \Phi(2\pi, X_n, \mu) - X_n$$

The actual construction of the vector function $\Psi(X_n, \mu)$ or the solution of (4.2) for $t = 2\pi$ can be effected by various approximate procedures. Let us assume that system (4.1) is completely integrable for $\mu = 0$. This enables us to use the small parameter method [4]. System of difference equations (1.3) no longer contains τ explicitly and is of the form

$$Z(\tau + \mu, \mu) - Z(\tau, \mu) = \mu\Psi(Z(\tau, \mu), \mu) \tag{4.8}$$

$$\Psi(Z, \mu) = \Psi_0(Z) + \mu\Psi_1(Z) + \mu^2\Psi_2(Z) + \ldots$$

The ancillary system of equations likewise does not contain τ explicitly,

$$dZ/d\tau = S(Z, \mu) \tag{4.9}$$

For $\tau = n\mu$ $(n = 0, 1, 2, \ldots)$ the vector $Z(\tau, \mu)$ assumes values which coincide with the values of X_n assumed by solution (4.2) of system (4.1) for $t = 2\pi n$. Replacing the independent variable in (4.9) according to the formula

$$2\pi\tau = \mu t \tag{4.10}$$

we arrive at the system of differential equations

$$\frac{dZ}{dt} = \frac{\mu}{2\pi} S(Z, \mu), \quad S(Z, \mu) = S_0(Z) + \mu S_1(Z) + \mu^2 S_2(Z) + \ldots \tag{4.11}$$

The solutions of systems (4.1) and (4.11) coincide for $t = 2n\pi$ $(n = 0, 1, 2, \ldots)$. The periodicity of system (4.1) implies that if the values of Z, t correspond to the values of X, t, then the values of Z, $t + 2\pi$ correspond to the values of X, $t + 2\pi$. Hence, X and Z are related by an expression periodic in t with the period 2π

$$X = P(t, Z, \mu), \quad P(t + 2\pi, Z, \mu) \equiv P(t, Z, \mu) \tag{4.12}$$
$$P(t, Z, \mu) = P_0(t, Z) + \mu P_1(t, Z) + \mu^2 P_2(t, Z) + \ldots$$

We can solve the equations of (4.12) for Z ,

$$Z = Q(t, X, \mu), \quad Q(t + 2\pi, X, \mu) \equiv Q(t, X, \mu) \tag{4.13}$$
$$Q(t, X, \mu) = Q_0(t, X) + \mu Q_1(t, X) + \mu^2 Q_2(t, X) + \ldots$$

We can find relations (4.12), (4.13) by integrating systems (4.1), (4.11). It is simpler, however, to attempt to find a substitution of the form (4.12) directly. This substitution reduces nonautonomous system of equations (4.1) to an autonomous system in standard form. For $\mu = 0$ we obtain

$$\frac{\partial P_0(t, Z)}{\partial t} = F(t, P_0(t, Z), 0), \quad P_0(t, Z) \equiv \Phi(t, Z, 0) \tag{4.14}$$

If system (4.1) is integrable for $\mu = 0$, then, substituting (4.12) and (4.11) into (4.1), we can determine $P_0, P_1, \ldots, S_0, S_1, \ldots$ successively. The asymptotic character of the resulting solutions follows from Theorem 1 [1].

Example 4.1. Let us find an approximate solution of the essentially nonlinear first-order differential equation

$$\frac{dx}{dt} = -x^2 \cos t - \mu x$$

by the asymptotic method. For $\mu = 0$ the equation is integrable and we have

$$x = \frac{c}{1 + c \sin t}, \qquad c = \frac{x}{1 - x \sin t}; \qquad x = c, \quad t = 0$$

We therefore seek our substitution (4.12) in the form

$$x = \frac{z}{1 + z \sin t} + \mu P_1(t, z) + \ldots, \qquad \frac{dz}{dt} = \mu S_0(z) + \ldots$$

Substituting these expressions into the differential equation, we obtain

$$s_0(z) + \frac{\partial}{\partial t}\left[(1 + z \sin t)^2 p_1(t, z)\right] = -z - z^2 \sin t$$

This yields expressions for $s_0(z)$, $p_1(t, z)$,

$$s_0(z) = -z, \qquad p_1(t, z) = \frac{z^2 \cos t}{(1 + z \sin t)^2}$$

Integrating, we obtain the approximate solution

$$x = \frac{c}{e^{\mu t} + c \sin t} + \mu \frac{c^2 \cos t}{(e^{\mu t} + c \sin t)^2} + O(\mu^2)$$

Note 4.1. For system (4.4) reduced to standard form substitution assumes the simpler form

$$X = Z + \mu P(t, Z, \mu), \qquad P(t + 2\pi, Z, \mu) \equiv P(t, Z, \mu) \tag{4.15}$$

since the solutions of system (4.4), (4.11) intersect at the instants $t = 2n\pi$ and since their derivatives are proportional to μ. We can immediately make the substitution

$$X = Z + \mu P_1(t, Z) + \mu^2 P_2(t, Z) + \mu^3 P_3(t, Z) + \ldots \tag{4.16}$$

which transforms nonautonomous system (4.4) into autonomous system (4.11). If we require that the solution $Z(t)$ of system (4.11) coincide with the solution of system (4.1) at the instants $t = 2n\pi$, then we can construct a unique ancillary system (4.11). The use of ancillary autonomous system (4.11) for asymptotic integration was first proposed by Krylov and Bogoliubov [5, 6] and elaborated by Mitropol'skii [3, 7]. Substitution (4.16) constructed jointly with system (4.11) leads to the asymptotic method of integration. Hence, for systems of the form (4.4) the asymptotic method of integration constitutes a particular, but practically convenient method of approximation in which the small-parameter method is applied to the method of point transformations. Neimark [8] pointed this out in the case of the first approximation of the asymptotic method called the "averaging method".

Note 4.2. The use of difference equations yields a method for the asymptotic integration of differential equations of the form

$$\frac{dX}{dt} = F(t, X, \mu, \mu t), \qquad F(t+2\pi, X, \mu, 0) \equiv F(t, X, \mu, 0) \tag{4.17}$$

A detailed investigation of Eqs. (4.17) containing the slow time μt was carried out by Mitropol'skii [7].

Note 4.3. All of the analytical methods of investigating nonlinear oscillations which we examined, e. g. the reservation method, the method of slowly varying coefficients (the Van der Pol method) [9], the method of equivalent linearization, the stroboscopic method of Minorsky [10, 11], the Poincaré method [4], etc., are reducible to the solution of nonlinear difference equations which are usually of the form (1.1). The use of the general method of point transformations [12] has not yet received sufficient elaboration for more complex cases. It leads to difference equations of the form

$$X_{n+1} = X_n + F(X_n, \mu n, \mu), \qquad F(X_n, \mu n, 0) \not\equiv 0$$

which will not be investigated here.

5. Obtaining periodic solutions. The periodic solutions of system (4.1) correspond to the fixed points of mapping (4.6). The initial vector X_0 satisfies the system of equations

$$\Psi(X_0, \mu) \equiv \Psi_0(X_0) + \mu\Psi_1(X_0) + \mu^2\Psi_2(X_0) + \ldots = 0 \tag{5.1}$$

The search for a periodic solution by asymptotic methods leads to ancillary system (4.11). The constant solutions of system (4.11) correspond to the periodic solutions of system (4.1). The initial vector X_0 which determines the periodic solution satisfies the system of equations

$$S(X_0, \mu) \equiv S_0(X_0) + \mu S_1(X_0) + \mu^2 S_2(X_0) + \ldots = 0 \tag{5.2}$$

Formulas (2.4)–(2.7) relating Ψ and S imply that Eqs. (5.1) and (5.2) are equivalent. For this reason, the periodic solutions provided by the asymptotic method and the method of Poincaré [4] can be made to coincide with any degree of accuracy desired. This was shown in [13] by direct computation for the first approximations. It is important to note that, having constructed Eqs. (5.2) defining the initial values for the periodic solution, we can construct ancillary system (4.11) without referring to system (4.1). By virtue of its uniqueness, system (4.11) can be arrived at by the asymptotic method [5–7], which makes it possible to investigate the stability of the periodic solutions. The Poincaré method can therefore be used to investigate transient processes. This has usually been done in the neighborhood of the fixed point [12]. Specifically, (2.3), (2.4) yield the approximate formula

$$S(Z, \mu) = 1.5\Psi(Z, \mu) - 0.5\Psi(Z + \mu\Psi(Z, \mu), \mu) + O(\mu^2) =$$
$$= \Psi(Z, \mu) - 0.5\mu \frac{D\Psi(Z, \mu)}{DZ}\Psi(Z, \mu) + O(\mu^2) \tag{5.3}$$

Example 5.1. Let us consider the stability of the solutions of the equation

$$x^{\cdot\cdot} + n^2 x = \mu F(t, x, x^{\cdot}, \mu), \qquad F(t + 2\pi, x, x^{\cdot}, \mu) \equiv F(t, x, x^{\cdot}, \mu) \tag{5.4}$$

in the first approximation in the resonance case $n = 1, 2, 3, \ldots$

Stipulating that

$$x = a, \ x^{\cdot} = b, \ t = 0$$

we arrive at the approximate solution

$$x(t) = x_0(t) + \frac{\mu}{n}\int_0^t F(\tau, x_0(\tau), x_0^{\cdot}(\tau), \mu) \sin n(t-\tau)\, d\tau + O(\mu^2)$$

$$x_0(t) \equiv a \cos nt + bn^{-1} \sin nt$$

Substituting in the value $t = 2\pi$ we obtain the new initial values

$$a_1 \equiv x(2\pi) = a + \mu P(a, b) + O(\mu^2), \qquad b_1 \equiv x^{\cdot}(2\pi) = b + \mu Q(a,b) = O(\mu^2)$$

where

$$P(a, b) = -\frac{1}{n}\int_0^{2\pi} F(\tau, x_0(\tau), x^{\cdot}_0(\tau), 0) \sin n\tau\, d\tau$$

$$Q(a, b) = \int_0^{2\pi} F(\tau, x_0(\tau), x_0^{\cdot}(\tau), 0) \cos n\tau\, d\tau$$

The difference equations for the coordinates of the point of intersection of the integral curve with the planes $t = 2n\pi$ are of the form

$$a_{n+1} - a_n = \mu P(a_n, b_n) + O(\mu^2), \qquad b_{n+1} - b_n = \mu Q(a_n, b_n) + O(\mu^2)$$

Hence, the differential equations for the slowly changing variables a, b assume the form (4.11),

$$\frac{da}{dt} = \frac{\mu}{2\pi} P(a, b) + O(\mu^2), \qquad \frac{db}{dt} = \frac{\mu}{2\pi} Q(a, b) + O(\mu^2) \tag{5.5}$$

If there exists a simple solution $a = a_0$, $b = b_0$ of the equations

$$P(a, b) = 0, \qquad Q(a, b) = 0$$

then Eq. (5.4) has a periodic solution. The stability of solutions in the first approximation is determined by the stability of system (5.5) linearized for $a = a_0$, $b = b_0$,

$$\frac{d(a-a_0)}{dt} = \frac{\mu}{2\pi}\frac{\partial P(a_0, b_0)}{\partial a_0}(a - a_0) + \frac{\mu}{2\pi}\frac{\partial P(a_0, b_0)}{\partial b_0}(b - b_0) + O(\mu^2)$$

$$\frac{d(b-b_0)}{dt} = \frac{\mu}{2\pi}\frac{\partial Q(a_0, b_0)}{\partial a_0}(a - a_0) + \frac{\mu}{2\pi}\frac{\partial Q(a_0, b_0)}{\partial b_0}(b - b_0) + Q(\mu^2)$$

The solutions are asymptotically stable if the roots of the characteristic equation

$$\begin{vmatrix} \partial P(a_0, b_0)/\partial a_0 - \lambda & \partial P(a_0, b_0)/\partial b_0 \\ \partial Q(a_0, b_0)/\partial a_0 & \partial Q(a_0, b_0)/\partial b_0 - \lambda \end{vmatrix} = 0 \tag{5.6}$$

have negative real parts. This conclusion was arrived at in [4], p. 80. The first approximation using difference equations actually coincides with the stroboscopic method of Minorsky [9, 10], who confines himself to the first approximation (as we are doing in this example) and usually converts to polar coordinates on the phase plane. Quite naturally, the first approximation of the asymptotic method [5–7] (usually called the "averaging method") leads to system (5.5), as does the Van der Pol method [11], p. 76.

6. The complex resonance case. Let us consider the quasilinear system

$$\frac{dX}{dt} = AX + \mu F(t, X, \mu), \qquad F(t + 2\pi, X, \mu) \equiv F(t, X, \mu) \tag{6.1}$$

We assume that all the solutions of the system are periodic with the period 2π for $\mu = 0$, i.e. that

$$\exp\{A2\pi\} = E \tag{6.2}$$

Solving system (6.1) with the initial vector X_0, we obtain the solution

$$X(t) = e^{At}X_0 + \mu\int_0^t e^{A(t-\tau)}F(\tau, e^{A\tau}X_0, \mu)\,d\tau + O(\mu^2) \tag{6.3}$$

System of difference equations (4.7) becomes

$$X_{n+1} - X_n = \mu\int_0^{2\pi} e^{-A\tau}F(\tau, e^{A\tau}X_n, 0)\,d\tau + O(\mu^2) \tag{6.4}$$

Equating the right side in (6.4) to zero, we obtain equations for finding the initial vector X_0 which determines the periodic solution. This periodic solution is asymptotically stable if the solution $Z = X_0$ of the system of differential equations

$$\frac{dZ}{dt} = \frac{\mu}{2\pi}\int_0^{2\pi} e^{-A\tau}F(\tau, e^{A\tau}Z, 0)\,d\tau \tag{6.5}$$

is stable.

This solution is in turn stable if all the eigenvalues of the matrix of the Jacobian

$$J = D\int_0^{2\pi} e^{-A\tau}F(\tau, e^{A\tau}Z, 0)\,d\tau/DZ \tag{6.6}$$

have negative real parts. If the matrix of (6.6) also has zero eigenvalues with elementary first-degree divisors, then the periodic solution of system (6.1) is stable in the first approximation. In doubtful cases the problem of stability can sometimes be resolved by considering system (6.5) directly.

Example 6.1. Let us investigate the stability of the periodic solution of the equation

$$x^{\cdot\cdot} + x + \mu x^2 \sin t = 0 \tag{6.7}$$

Equations (5.5) are of the form

$$\frac{da}{dt} = \frac{\mu}{8}(a^2 + 3b^2) + O(\mu^2), \qquad \frac{db}{dt} = -\frac{\mu}{4}ab + O(\mu^2) \tag{6.8}$$

We obtain $a_0 = 0$, $b_0 = 0$ for the periodic solution. Equation (5.6) has a multiple zero root. Application of the small-parameter method in its usual form [4] requires computation of subsequent approximations. On the other hand, it is sufficient to consider the integral curves of system (6.8) on the plane ab to conclude that the solution $a = 0$, $b = 0$ of system (6.8) (i.e. the periodic solution $x = 0$ of Eq. (6.7)) is unstable. The solution $x = 0$ stable in the first approximation is unstable when subsequent approximations are taken into account.

7. The integral stability criterion (in a form different from that used in [14–16]). Let us consider the oscillations of a system described by the generalized coordinates $q_1, \ldots, q_n$. We assume for simplicity that conversion to the principal coordinates is possible for the unperturbed system. The kinetic potential is of the form

$$L(q_i, q_i^{\cdot}, \omega t, \mu) \equiv T - \Pi = \frac{1}{2}\sum_{i=1}^{n}(q_i^{\cdot 2} - \omega_i^2 q_i^2) + \mu l_1(q_i, q_i^{\cdot}, \omega t, \mu) \tag{7.1}$$

Let us assume that $\mu > 0$, where μ is a small parameter, and that the periodicity condition

$$l(q_i, q_i^{\cdot}, \Theta + 2\pi, \mu) \equiv l(q_i, q_i^{\cdot}, \Theta, \mu)$$

is fulfilled.

We shall consider the nearly-resonance case when the perturbation frequency ω and the proper frequencies ω_i of the unperturbed system ($\mu = 0$) are in rational ratio. We assume that

$$\frac{\omega_i}{\omega} \approx \frac{p_i}{N}, \quad \omega_i^2 - \nu_i^2 = O(\mu), \quad \nu_i = \frac{p_i \omega}{N} \quad (i = 1, \ldots, n) \tag{7.2}$$

Here the p_i are nonnegative integers and N is a sufficiently large positive number. We obtain the following expression for the kinetic potential:

$$L(q_i, q_i^{\cdot}, \omega t, \mu) = \frac{1}{2} \sum_{i=1}^{n} (q_i^{\cdot 2} - \nu_i^2 q_i^2) + \mu l(q_i, q_i^{\cdot}, \omega t, \mu) \tag{7.3}$$

$$l(q_i, q_i^{\cdot}, \omega t, \mu) \equiv l_1(q_i, q_i^{\cdot}, \omega t, \mu) + \mu^{-1} \sum_{i=1}^{n} \frac{q_i^2}{2} (\nu_i^2 - \omega_i^2)$$

All of the solutions of the unperturbed system ($\mu = 0$) with the kinetic potential L (7.3) are periodic with the common period $T = 2\pi N \omega^{-1}$. The Lagrange differential equations now become

$$q_i^{\cdot\cdot} + \nu_i^2 q_i = -\mu \left[\frac{d}{dt} \frac{\partial l}{\partial q_i^{\cdot}} - \frac{\partial l}{\partial q_i} \right], \quad \nu_i = \frac{p_i \omega}{N} \quad (i = 1, 2, \ldots, n) \tag{7.4}$$

In the zeroth approximation we obtain the generating solution

$$q_{i0}(t) = a_i \cos \nu_i t + (b_i / \nu_i) \sin \nu_i t \tag{7.5}$$

Here a_i, b_i are the initial values of $q_i(t)$, $q_i^{\cdot}(t)$ for $t = 0$. Let us compute the new initial values a_{i1}, b_{i1} after the period T. Applying the small parameter method, we obtain

$$q_i(t) = q_{i0}(t) - \frac{\mu}{\nu_i} \int_0^t \left[\frac{d}{dt} \frac{\partial l}{\partial q_i^{\cdot}} - \frac{\partial l}{\partial q_i} \right]_0 \sin \nu_i (t - \tau)\, d\tau + O(\mu^2) \tag{7.6}$$

The expression in square brackets in the integrand is to be computed for the generating solution $q_{i0}(\tau)$, $q_{i0}^{\cdot}(\tau)$. We indicate this by means of the subscript $_0$. We now have the point transformation (7.7)

$$a_{i1} = a_i + \mu T P_i(a_j, b_j) + O(\mu^2), \quad b_{i1} = b_i + \mu T Q_i(a_j, b_j) + O(\mu^2)$$

where

$$P_i(a_j, b_j) = \frac{1}{\nu_i T} \int_0^T \left[\frac{d}{d\tau} \frac{\partial l}{\partial q_i^{\cdot}} - \frac{\partial l}{\partial q_i} \right]_0 \sin \nu_i \tau\, d\tau, \quad T = \frac{2\pi N}{\omega} \tag{7.8}$$

$$Q_i(a_j, b_j) = -\frac{1}{T} \int_0^T \left[\frac{d}{d\tau} \frac{\partial l}{\partial q_i^{\cdot}} - \frac{\partial l}{\partial q_i} \right]_0 \cos \nu_i \tau\, d\tau, \quad \mu = 0$$

Integration by parts makes the extraintegral terms vanish because of the periodicity of the integrands, and we have

$$P_i(a_j, b_j) = -\frac{1}{T} \int_0^T \left[\frac{\partial l}{\partial q_i^{\cdot}} \cos \nu_i \tau + \frac{\partial l}{\partial q_i} \frac{\sin \nu_i \tau}{\nu_i} \right]_0 d\tau = -\frac{1}{T} \int_0^T \left[\frac{\partial l}{\partial b_i} \right]_0 d\tau \tag{7.9}$$

$$Q_i(a_j, b_j) = -\frac{1}{T} \int_0^T \left[\frac{\partial l}{\partial q_i^{\cdot}} \nu_i \sin \nu_i \tau + \frac{\partial l}{\partial q_i} \cos \nu_i \tau \right]_0 d\tau = \frac{1}{T} \int_0^T \left[\frac{\partial l}{\partial a_i} \right]_0 d\tau$$

Let us introduce the ancillary function

$$\Lambda(a_j, b_j) = \frac{1}{T}\int_0^T L(q_{i0}(\tau), q_{i0}^{\cdot}(\tau), \omega\tau)\,d\tau = \frac{\mu}{T}\int_0^T [l]_0\,d\tau \qquad (7.10)$$

The above equation is fulfilled, since the relation

$$\int_0^T \sum_{i=1}^n [q_{i0}^{\cdot 2}(\tau) - \nu_i^2 q_{i0}^2(\tau)]_0\,d\tau \equiv 0 \qquad (7.11)$$

is fulfilled for the generating solution.

Here $\Lambda\ (a_j,\ b_j)$ is the average value of the kinetic potential L of the perturbed system computed for unperturbed solution (7.5). Since N in T can be arbitrarily large, we can assume that

$$\Lambda(a_j, b_j) = \lim_{T\to\infty}\frac{1}{T}\int_0^T L(q_{i0}(\tau), q_{i0}^{\cdot}(\tau), \omega\tau, \mu)\,d\tau, \quad \Lambda = O(\mu) \qquad (7.12)$$

As our generating solution $q_{i0}\ (\tau)$ we take the expressions defined by formulas (7.5). These define harmonic oscillations with frequencies ν_i rationally commensurate with the perturbation frequency ω. Mapping (7.7) becomes

$$a_{i1} = a_i - T\frac{\partial\Lambda}{\partial b_i} + O(\mu^2), \qquad b_{i1} = b_i + T\frac{\partial\Lambda}{\partial a_i} + O(\mu^2) \qquad (7.13)$$

The corresponding system of differential equations (4.11) is canonical in the first approximation,

$$\frac{da_i}{dt} = -\frac{\partial\Lambda}{\partial b_i} + O(\mu^2), \quad \frac{db_i}{dt} = \frac{\partial\Lambda}{\partial a_i} + O(\mu^2) \qquad (7.14)$$

The equations for determining the periodic solution in the first approximation are

$$\frac{\partial\Lambda(a_j, b_j)}{\partial a_i} = 0, \quad \frac{\partial\Lambda(a_j, b_j)}{\partial b_i} = 0 \quad (i = 1, \ldots, n) \qquad (7.15)$$

Hence, the initial values $a_{i0},\ b_{i0}$ which determine the periodic solution are the coordinates of the fixed point for the function $\Lambda\ (a_j,\ b_j)$. Equations (7.14) have the energy integral

$$\Lambda(a_j, b_j) = \text{const} + O(\mu^2) \qquad (7.16)$$

The surfaces with Eq. (7.16) are closed in the neighborhood of the fixed point $a_{i0},\ b_{i0}$ in the phase space of the variables $a_j,\ b_j$ if the function $\Lambda\ (a_j,\ b_j)$ has either a minimum or a maximum at this point. Otherwise surfaces (7.16) are not closed.

Theorem. (The integral stability criterion). Let us compute the average value $\Lambda\ (a_j,\ b_j)$ of the kinetic potential of the perturbed system ($\mu \neq 0$) along the periodic solution of the unperturbed system ($\mu = 0$) as a function of the initial values $a_i,\ b_i$. If the function Λ has either a maximum or a minimum at the point $a_{i0},\ b_{i0}$, then this point determines the periodic solution stable in the first approximation. The other fixed points require special consideration.

Example 7.1. Let us consider the system with the kinetic potential

$$L(x,\ x^{\cdot},\ t) = {}^1/_2\ (x^{\cdot 2} - x^2 - \mu\lambda x^2) + \mu x x^{\cdot} \sin 2t$$

The corresponding differential equation is a Mathieu linear differential equation,

$$x^{\cdot\cdot} + (1 + \mu\lambda + 2\mu\cos 2t)x = 0 \qquad (7.17)$$

For $\mu = 0$ this equation has the solution

$$x_0(t) = a\cos t + b\sin t$$

Let us compute the average value of the kinetic potential for this solution,

$$\Lambda \equiv \frac{1}{2\pi}\int_0^{2\pi} L(x_0(\tau), x_0^{\cdot}(\tau), \tau)\, d\tau = \frac{\mu}{4}[-(\lambda+1)a^2 + (1-\lambda)b^2]$$

At the fixed point $a = 0$, $b = 0$ the function $\Lambda(a, b)$ has a minimum for $\lambda < -1$ and a maximum for $\lambda > 1$. For $-1 < \lambda < 1$ the point $a = 0$, $b = 0$ is a saddle point. Hence, for $|\lambda| > 1$ we have a stable zero solution of Eq. (7.17) and for $|\lambda| > 1$ an unstable zero solution. For $\lambda = \pm 1$ the fixed point is defined ambiguously, which implies the existence of a family of periodic solutions. The implications of this example agree with the known results of [17].

Note 7.1. The integral criterion can be broadened considerably to cover systems canonical in the zeroth approximation. It bears a close relationship to perturbation theory [18]. Let us formulate one of the implications without proof or elaboration. We consider a canonical system of differential equations with the Hamiltonian H,

$$H \equiv H(t, q_j, p_j, \mu) \tag{7.18}$$

which is almost periodic in t and doubly differentiable with respect to all of its arguments. Let the generating canonical system

$$q_s^{\cdot} = \frac{\partial H_0}{\partial p_s}, \quad p_s^{\cdot} = -\frac{\partial H_0}{\partial q_s}, \quad H_0 = H(t, q_j, p_j, 0) \quad (s = 1, \ldots, n) \tag{7.19}$$

have the generating solution

$$q_s = q_{s0}^{\cdot}(t, a_j, b_j), \qquad p_s = p_{s0}^{\cdot}(t, a_j, b_j) \tag{7.20}$$

where a_j, b_j are the initial values of q_j, p_j for $t = 0$. Taking a_j, b_j as our new variables and assuming that the averaging method [19] is applicable to the system of equations

$$\frac{da_s}{dt} = \frac{\partial(H - H_0)}{\partial b_s}, \qquad \frac{db_s}{dt} = -\frac{\partial(H - H_0)}{\partial a_s} \qquad (s = 1, \ldots, n) \tag{7.21}$$

we conclude that the solution of the system with the Hamiltonian H (7.18) is stable with respect to the parameters a_j, b_j, if the function $\Lambda(a_j, b_j)$,

$$\Lambda(a_j, b_j) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\sum_{s=1}^n p_{s0} q_{s0}^{\cdot} - H(t, q_{j0}, p_{j0}, \mu)\right] dt \tag{7.22}$$

has a minimum or a maximum for these values of a_j, b_j.

The stability criterion is also applicable to the analysis of the stability of almost-periodic solutions. The parameters can be arbitrary constants which do not necessarily coincide with the initial conditions.

Example 7.2. Let us use the integral criterion to find the stability condition under combination resonance of the solutions of the system

$$q_1^{\cdot\cdot} + \omega_1^2 q_1 - 2\mu q_2 \cos\omega t = 0, \quad q_2^{\cdot\cdot} + \omega_2^2 q_2 - 2\mu q_1 \cos\omega t = 0, \quad \omega \approx \omega_1 + \omega_2$$

The system is canonical with the Hamiltonian

$$H(q_j, p_j, t) = {}^1\!/_2 (p_1^2 + \omega_1^2 q_1^2 + p_2^2 + \omega_2^2 q_2^2) + 2\mu q_1 q_2 \cos\omega t$$

We choose our generating solution

$$q_1 = A_1 \cos(\nu_1 t + \alpha_1), \quad q_2 = A_2 \cos(\nu_2 t + \alpha_2), \quad \omega_1 - \nu_1 = O(\mu), \quad \omega_2 - \nu_2 = O(\mu)$$

such that the exact relation

$$\nu_1 + \nu_2 = \omega, \qquad \omega = \omega_1 + \omega_2 - (\omega_1 - \nu_1) - (\omega_2 - \nu_2)$$

is fulfilled.

The average value of the kinetic potential is of the form

$$\Lambda (A_j, \alpha_j) = {}^1/_4 \,[A_1^2(\omega_1^2 - \nu_1^2) + 2\mu A_1 A_2 \cos(\alpha_1 + \alpha_2) + A_2^2(\omega_2^2 - \nu_1^2)]$$

If

$$(\omega_1^2 - \nu_1^2)\ (\omega_2^2 - \nu_2^2) > \mu^2$$

then a maximum or a minimum occurs at the point $A_1 = A_2 = 0$ corresponding to the zero solution.

The above condition can be simplified into

$$4\omega_1\omega_2(\omega_1 - \nu_1)\ (\omega_2 - \nu_2) > \mu^2 + O(\mu^3)$$

Varying ν_1, ν_2, we obtain the largest stability domain for

$$\omega_1 - \nu_1 = \omega_2 - \nu_2$$

The stability domain is defined by the inequality

$$|\omega - \omega_1 - \omega_2| > \frac{\mu}{\sqrt{\omega_1\omega_2}}$$

Note 7.3. We note that this result is correct despite the fact that the averaging method is not directly applicable [20] to Eqs. (7.21) obtained in the course of its derivation. We refer the reader to interesting paper [21] , which concerns averaging in canonical systems. The example is analyzed by another method in [22].

8. Stability of systems with friction. Let the system considered in Sect. 7 be acted on by friction with the Rayleigh dispersion function

$$F = {}^1/_2\mu(\varepsilon_1 q_1^{\cdot 2} + \varepsilon_2 q_2^{\cdot 2} + \ldots + \varepsilon_n q_n^{\cdot 2}), \quad \varepsilon_j \geqslant 0 \tag{8.1}$$

Equations (7.4) now become

$$q_i^{\cdot\cdot} + \mu\varepsilon_i q_i^{\cdot} + \nu_i^2 q_i = -\mu\left[\frac{d}{dt}\frac{dl}{\partial q_i^{\cdot}} - \frac{dl}{\partial q_i}\right] \tag{8.2}$$

Making substitution (7.5) and converting first to difference equations and then to differential equations, we obtain the first-approximation equations

$$\frac{da_i}{dt} = -\frac{\partial\Lambda}{\partial b_i} - \frac{\partial R}{\partial a_i}, \qquad \frac{db_i}{dt} = \frac{\partial\Lambda}{\partial a_i} - \frac{\partial R}{\partial b_i} \tag{8.3}$$

where

$$\Lambda(a_j, b_j) = \lim_{T\to\infty}\frac{1}{T}\int_0^T L(q_{i0}(\tau), q_{i0}^{\cdot}(\tau), \omega\tau, \mu)\, d\tau, \qquad R(a_j, b_j) = \frac{\mu}{4}\sum_{i=1}^{n}\varepsilon_i(a_i^2 + b_i^2) \tag{8.4}$$

Equations (8.3) enable us to introduce the transient processes. In order to obtain the almost-periodic or periodic solutions we must equate the right sides of (8.3) to zero. To investigate the stability of the resulting a_{j0}, b_{j0} in the first approximation, we introduce the matrices P, Q, R, L with elements computed at the point a_{j0}, b_{j0}

$$p_{ks} = \frac{\partial^2\Lambda}{\partial a_k\partial a_s}, \quad q_{ks} = \frac{\partial^2\Lambda}{\partial a_k\partial b_s}, \quad r_{ks} = \frac{\partial^2\Lambda}{\partial b_k\partial b_s}, \quad l_{ks} = \frac{\mu\varepsilon_k}{2}\delta_{ks} \tag{8.5}$$

Here δ_{ks} is the Kronecker delta: $\delta_{kk} = 1$, $\delta_{ks} = 0$ $(k \neq s)$.

The characteristic equation of the system in variations is of the form

$$\mathrm{Det}\begin{Vmatrix} \lambda E + L + Q' & R \\ -P & \lambda E + L - Q \end{Vmatrix} = 0 \tag{8.6}$$

In this block matrix E represents an identity matrix and Q' the matrix adjoint to Q. In the case of simple resonances when only one of the characteristic frequencies of the system for $\mu = 0$ is equal to the double perturbation frequency, the necessary and sufficient condition of asymptotic stability follows from the inequality

$$\mathrm{Det}\begin{Vmatrix} P & Q-L \\ Q'+L & R \end{Vmatrix} > 0 \tag{8.7}$$

Example 8.1. Let us investigate the stability of oscillations with the kinetic potential L, $L = \frac{1}{2}(x^{\cdot 2} + y^{\cdot 2} - \omega_1^2 x^2 - \omega_2^2 y^2) + \mu c x^2 y + yq \sin \omega t, \quad \omega \approx 2\omega_1$
and the Rayleigh dispersion function

$$F = \tfrac{1}{2}\mu(\varepsilon_1 x^{\cdot 2} + \varepsilon_2 y^{\cdot 2}), \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0$$

We obtain the following differential equations of the zeroth approximation:

$$x^{\cdot\cdot} + 0.25\omega^2 x = 0, \qquad y^{\cdot\cdot} + \omega_2^2 y = q \sin \omega t$$

As our generating solution we take the expressions

$$x = a_1 \cos \frac{\omega}{2} t + \frac{2b_1}{\omega} \sin \frac{\omega}{2} t, \quad y = a_2 \cos \omega_2 t + \frac{b_2}{\omega_2} \sin \omega_2 t + \frac{q}{\omega_2^2 - \omega^2} \sin \omega t$$

From formula (8.4) for $\omega_2 \neq \omega_1$, $2\omega_2 \neq \omega_1$, $\omega_2 \neq 0$ we obtain

$$\Lambda = \frac{a_1^2}{2}\left(\frac{\omega^2}{4} - \omega_1^2\right) + \frac{b_1^2}{2}\left(1 - \frac{4\omega_1^2}{\omega^2}\right) + \frac{\mu c a_1 b_1 q}{2\omega(\omega_2^2 - \omega^2)}$$

Stability condition (8.7) becomes

$$\frac{\mu^2\varepsilon_2^2}{4}\left[\frac{(\omega^2 - 4\omega_1^2)}{4\omega^2} + \frac{\mu^2\varepsilon_1^2}{4} - \frac{\mu^2 c^2 q^2}{4\omega^2(\omega_2^2 - \omega^2)}\right] > 0$$

After some simplifying operations it becomes the inequality

$$4(\omega - 2\omega_1)^2 + \mu^2\varepsilon_1^2 - \frac{\mu^2 c^2 q^2}{\omega^2(\omega_2^2 - \omega^2)} > 0$$

As is clear from this example, it is not necessary to construct the complete differential equations of motion in order to investigate stability. This makes the integral criterion a convenient means of analyzing the stability of the oscillations of complex mechanical systems.

9. The canonical difference equations. Let us apply the asymptotic method [5–7] to the canonical system of differential equations with the small parameter μ,

$$q_s^{\cdot} = \mu \frac{\partial H}{\partial p_s}, \quad p_s^{\cdot} = -\mu \frac{\partial H}{\partial q_s}, \quad H = H(q_i, p_j, t) \quad (s = 1, \ldots, n) \tag{9.1}$$

where the Hamiltonian H is differentiable a sufficient number of times and is 2π-periodic in t.

We begin by introducing the general solution of system (9.1),

$$q_s = \varphi_s(t, a_j, b_j, \mu), \quad p_s = \psi_s(t, a_j, b_j, \mu) \quad (s = 1, \ldots, n) \tag{9.2}$$

with the initial conditions

$$q_s = a_s, \quad p_s = b_s, \quad t = 0 \quad (s = 1, \ldots, n)$$

Next we find the mapping given by solution (9.2) after the period 2π,

$$a_{s,k+1} = \Phi_s(a_{jk}, b_{jk}, \mu), \quad b_{s,k+1} = \Psi_s(a_{jk}, b_{jk}, \mu) \quad (k = 0, 1, 2, \ldots) \tag{9.3}$$

$$\Phi_s(a_j, b_j, \mu) \equiv \varphi_s(2\pi, a_j, b_j, \mu), \quad \Psi_s(a_j, b_j, \mu) \equiv \psi_s(2\pi, a_j, b_j, \mu)$$

Since system (9.1) is canonical, we have the relative integral invariant (see [23], p. 302)

$$\int_{C_{k+1}} \sum_{s=1}^{n} \Psi_s \delta\Phi_s = \int_{C_k} \sum_{s=1}^{n} b_{sk}\delta a_{sk} \tag{9.4}$$

Here C_k is a closed contour in $2n$-dimensional space obtainable from C_0 by means of

k successive mappings (9.3). Difference equations (9.3) satisfying condition (9.4) will be called "canonical" equations. Let us introduce an ancillary system of differential equations of the form (4.9) for difference equations (9.3). Its solution for $\tau = k\mu$ ($k = 0, 1, 2, \ldots$) has a relative integral invariant. Taking the limit as $\mu \to 0$, we infer from this that the ancillary system has a relative integral invariant of the form

$$\int \sum_{k=1}^{n} p_k \delta q_k$$

and is therefore canonical. Making substitution (4.10), we obtain

$$u_s^{\cdot} = \mu \frac{\partial H_1(u_j, v_j, \mu)}{\partial v_s}, \quad v_s^{\cdot} = -\mu \frac{\partial H_1(u_j, v, \mu)}{\partial u_s} \quad (s = 1, \ldots, n) \tag{9.5}$$

This means that there exists a canonical transformation of variables

$$q_s = u_s + \sum_k \mu^k U_k(u_j, v_j, t), \quad p_s = v_s + \sum_k \mu^k V_k(u_j, v_j, t) \tag{9.6}$$

which transforms canonical system (9.1) into canonical autonomous system (9.5). Thus, by applying the asymptotic method to the canonical system we can obtain the ancillary autonomous system in canonical form. This fact is proved in a different way in [24].

Note 9.1. It is possible to seek a function $W(q_j, u_j, t, \mu)$ which determines the canonical transformation in accordance with the formulas

$$v_s = \frac{\partial W}{\partial u_s}, \quad p_s = -\frac{\partial W}{\partial q_s} \quad (s = 1, \ldots, n) \tag{9.7}$$

and is such that the function H_1 does not contain the time t,

$$H_1 = H - \frac{\partial W}{\partial t}, \quad \frac{\partial H_1}{\partial t} \equiv 0 \tag{9.8}$$

10. Complex resonance in autonomous systems. Let us consider the system with $n + 1$ degrees of freedom

$$x_i^{\cdot\cdot} + \omega_i^2 x_i = \mu f_i(x_j, x_j^{\cdot}) \quad (i = 0, 1, \ldots, n), \quad \omega_i \neq 0 \tag{10.1}$$

where the ratio of any two frequencies ω_i is a rational number. By replacing the independent variable we can reduce system (10.1) to the case where all the ω_i are positive integers with the largest common divisor equal to unity. Let us find the solution under the initial conditions

$$x_0 = u, \quad x_0^{\cdot} = 0, \quad x_i = y_i, \quad x_i^{\cdot} = z_i \quad (i = 1, \ldots, n), \ t = 0 \tag{10.2}$$

For $\mu = 0$ all the solutions of (10.1) are periodic with the period 2π. Let us find (for $\mu > 0$) a mapping which is effected along the trajectories of system (10.1) from the instant $t = 0$ to an instant t^* close to 2π at which $x_0^{\cdot} = 0$. It is more convenient in practice to find the preliminary values of the variables at the instant 2π. Knowing that for $\mu = 0$ in the zeroth approximation we have

$$x_{00}(t) = u \cos \omega_0 t, \quad x_{i0}(t) = y_i \cos \omega_i t + \frac{z_i}{\omega_i} \sin \omega_i t \quad (i = 1, \ldots, n) \tag{10.3}$$

we can apply the small-parameter method to obtain

$$x_i(t) = x_{i0}(t) + \mu x_{i1}(t) + O(\mu^2) \quad (i = 0, 1, \ldots, n) \tag{10.4}$$

$$x_{i1}(t) = \frac{1}{\omega_i} \int_0^t f_i(x_{j0}(\tau), x_{j0}^{\cdot}(\tau)) \sin \omega_i (t - \tau) \, d\tau$$

Using the ellipsis to denote terms of order μ^2 and higher, we obtain the following values for $t = 2\pi$:

$$x_0(2\pi) = u + \mu x_{01}(2\pi) + \ldots, \quad x_0^{\cdot}(2\pi) = \mu x_{01}^{\cdot}(2\pi) + \ldots \quad (10.5)$$
$$x_i(2\pi) = y_i + \mu x_{i1}(2\pi) + \ldots, \quad x_i^{\cdot}(2\pi) = z_i + \mu x_{i1}^{\cdot}(2\pi) + \ldots$$
$$(i = 1, \ldots, n)$$

Now let us move along the trajectories to the instant t^* at which $x_0^{\cdot}(t^*) = 0$. To this end we convert from (10.1) to the first-order system

$$x_i^{\cdot} = u_i, \quad u_i^{\cdot} = -\omega_i^2 x_i + \mu f_i(x_j, u_j)$$

Let us take u_0 as the independent variable. This yields the system

$$\frac{dx_i}{du_0} = \frac{u_i}{-\omega_0^2 x_0 + \mu f_0(x_j, u_j)}, \qquad \frac{du_i}{du_0} = \frac{-\omega_i^2 x_i + \mu f_i(x_j, u_j)}{-\omega_0^2 x_0 + \mu f_0(x_j, u_j)} \quad (10.6)$$

Solving this approximately under initial conditions (10.5) up to the value $u_0 = 0$, we obtain

$$x_0(t^*) = u + \mu x_{01}(2\pi) + \ldots, \quad x_0^{\cdot}(t^*) = 0, \quad t^* = 2\pi + \frac{\mu}{\omega_0^2 u} x_{01}^{\cdot}(2\pi) + \ldots \quad (10.7)$$
$$x_i(t^*) = y_i + \mu\left[x_{i1}(2\pi) + \frac{z_i}{\omega_0^2 u} x_{01}^{\cdot}(2\pi)\right] + \ldots$$
$$x_i^{\cdot}(t^*) = z_i + \mu\left[x_{i1}^{\cdot}(2\pi) - \frac{\omega_i^2 z_i}{\omega_0^2 u} x_{01}^{\cdot}(2\pi)\right] + \ldots$$

Formulas (10.7) define the point transformation of the values of u, y_i, z_i into the analogous values at the instant t^* which itself depends on u, y_i, z_i. Let us convert to an ancillary system of differential equations of the form (4.11). From (10.4), (10.7) we obtain the first-approximation equations

$$\frac{du}{ds} = -\frac{\mu}{2\pi\omega_0}\int_0^{2\pi}[f_0]\sin\omega_0\tau\, d\tau + O(\mu^2)$$

$$\frac{dy_i}{ds} = -\frac{\mu}{2\pi}\int_0^{2\pi}\left\{[f_i]\frac{\sin\omega_i\tau}{\omega_i} - [f_0]\frac{z_i\cos\omega_0\tau}{\omega_0^2 u}\right\}d\tau + O(\mu^2) \quad (10.8)$$

$$\frac{dz_i}{ds} = \frac{\mu}{2\pi}\int_0^{2\pi}\left\{[f_i]\cos\omega_i\tau - [f_0]\frac{y_i\omega_i^2}{u\omega_0^2}\cos\omega_0\tau\right\}d\tau + O(\mu^2)$$

In this fashion we have replaced the system of initial order $2n + 1$ by an autonomous system of order $2n + 2$. The new system can be conveniently handled by approximate methods by virtue of the small factor in its right sides. The variable s denotes the local time along each trajectory. It is related to t by the differential equation

$$\frac{dt}{ds} = 1 - \frac{\mu}{2\pi\omega_0 u}\int_0^{2\pi}[f_0]\cos\omega_0\tau\, d\tau + O(\mu^2) \quad (10.9)$$

The expressions in square brackets are computed for generating solution (10.3).

The periodic solution can be found by equating the right sides of system (10.8) to zero. We shall investigate stability by means of equations in variations, which in this case are linear differential equations with constant coefficients.

E x a m p l e 10.1. For the system of differential equations

$$x_0^{\cdot\cdot} + n^2 x_0 = \mu x_1^2 x_0^{\cdot}, \quad x_1^{\cdot\cdot} + m^2 x_1 = -\mu x_0^2 x_1^{\cdot}, \quad \mu > 0, \quad m \neq n$$

Eqs. (10.8), (10.9) become

$$\frac{du}{dt}=\frac{\mu u}{4}(y^2+z^2m^{-2}),\quad \frac{ds}{dt}=1+O(\mu^2),\quad \frac{dy}{dt}=-\frac{\mu}{4}yu^2,\quad \frac{dz}{dt}=-\frac{\mu}{4}zu^2$$

The above system becomes readily integrable once we have made the substitutions

$$p=u^2,\quad q=y^2+z^2m^{-2},\quad \frac{dp}{dt}=\frac{\mu}{2}pq,\quad \frac{dq}{dt}=-\frac{\mu}{2}pq$$

Without solving the equations we note that there are two families of periodic solutions, the stable family

$$x_0=0,\quad x_1=y\cos mt+zm^{-1}\sin mt,\quad p=0$$

and the unstable family

$$x_0=u\cos nt,\quad x_1=0,\quad q=0$$

All of the solutions become the stable periodic solution in such a way that

$$p+q=C,\quad u^2+y^2+z^2m^{-2}=C$$

The equations yield the approximate solution

$$x_0(t)=C_1(1+C_2e^{0.5\mu C_1^2t})^{-1/2}\cos(nt+C_3)$$

$$x_1(t)=C_1(1+C_2^{-1}e^{-0.5\mu C_1^2t})^{-1/2}\sin(mt+C_4)$$

containing four arbitrary constants.

Resonance in system of differential equations (10.1) facilitates their analysis.

BIBLIOGRAPHY

1. Valeev, K. G., On the theory of the point transformation method. Izv. VUZ, Radiofizika Vol. 11, №7, 1968.
2. Arnol'd, V. I., A proof of Kolmogorov's theorem on the preservation of nominally periodic motions with a small change in the Hamiltonian, Usp. Mat. Nauk Vol. 18, №5, 1963.
3. Bogoliubov, N. N. and Mitropol'skii, Iu. A., Asymptotic Methods in the Theory of Nonlinear Oscillations. Moscow, Fizmatgiz, 1963.
4. Malkin, I. G., Some Problems of the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
5. Krylov, N. M. and Bogoliubov, N. N., Applications of the Methods of Nonlinear Mechanics in Steadystate Oscillation Theory. Kiev, Izd. Akad. Nauk USSR, 1934.
6. Krylov, N. M. and Bogoliubov, N. N., Introduction to Nonlinear Mechanics. Kiev, Izd. Akad. Nauk USSR, 1937.
7. Mitropol'skii, Iu. A., Problems of the Asymptotic Theory of Unsteady Oscillations. Moscow, "Nauka", 1964.
8. Neimark, Iu. I., The averaging method from the standpoint of the point transformation method. Izv. VUZ, Radiofizika Vol. 6, №5, 1963.
9. Kauderer, G., Nonlinear Mechanics. (Russian translation). Moscow, Izd. Inostr. Lit., 1961.
10. Minorsky, N., Sur la méthode stroboscopique. Compt. Rend., Acad. Sci. Vol. 256, №19, 1963.
11. Butenin, N. V., Elements of the Theory of Nonlinear Oscillations. Leningrad, Sudpromgiz, 1962.

12. Neimark, Iu. I., The point transformation method in the theory of nonlinear oscillations. In: Proceedings of the International Symposium on Nonlinear Oscillations. Vol. 2, Kiev, Izd. Akad. Nauk USSR, 1963.
13. Proskuriakov, A. P., Comparison of the periodic solutions of quasilinear systems constructed by the method of Poincaré and by the method of Krylov-Bogoliubov. PMM Vol. 28, №4, 1964.
14. Blekhman, I. I. and Lavrov, B. P., On an integral criterion for stability of motion. PMM Vol. 24, №5, 1960.
15. Blekhman, I. I., Justification of the integral criterion of stability of motion in problems on autosynchronization of vibrators. PMM Vol. 24, №6, 1960.
16. Nagaev, R. F., Synchronization in a system essentially nonlinear objects with a single degree of freedom. PMM Vol. 29, №2, 1965.
17. McLaughlan, N. W., The Theory and Applications of Mathieu Functions. (Russian translation). Izd. Inostr. Lit., 1953.
18. Lur'e, A. I., Analytical Mechanics. Moscow, Fizmatgiz, 1961.
19. Mitropol'skii, Iu. A., Lectures on the Averaging Method in Nonlinear Mechanics. Kiev, "Naukova dumka", 1966.
20. Hsu, C. S., On the parametric excitation of a dynamic system having multiple degrees of freedom. Trans. ASME J. Appl. Mech. Vol. 30, №3, 1963.
21. Musa Samuel, A., Integral constraints in weakly nonlinear periodic systems. SIAM J. Appl. Math. Vol. 15, №5, 1967.
22. Valeev, K. G., On Hill's method in the theory of linear differential equations with periodic coefficents. Determination of the characteristic exponents. PMM Vol. 25, №2, 1961.
23. Whittaker, E. T., Analytical Dynamics. (Russian translation). Moscow-Leningrad, 1937.
24. Fedorchenko, A. M., The canonical averaging method in the theory of nonlinear oscillations. Ukr. Mat. Zhurnal Vol. 9, №2, 1957.

Translated by A. Y.

ON THE MOTION OF A HOLLOW BODY FILLED WITH VISCOUS LIQUID ABOUT ITS CENTER OF MASS IN A POTENTIAL BODY-FORCE FIELD

PMM Vol. 33, №3, 1969, pp. 431-440

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(Received December 2, 1968)

We consider the motion of a hollow solid body whose cavity is completely filled with a viscous liquid, assuming that the product of the Reynolds and Strouhal characteristic numbers for the flow of the viscous fluid in the cavity is small. We then show that the problem can be handled by methods used to investigate systems with a small parameter accompanying the higher derivatives and develop an algorithm for constructing an asymptotic expansion of the corresponding simultaneous system of Navier-Stokes and ordinary